

# POINTWISE RATES OF CONVERGENCE FOR THE OLIKER-PRUSSNER METHOD FOR THE MONGE-AMPÈRE EQUATION

RICARDO H. NOCHETTO AND WUJUN ZHANG

**ABSTRACT.** We study the Oliker-Prussner method exploiting its geometric nature. We derive discrete stability and continuous dependence estimates in the max-norm by using a discrete Alexandroff estimate and the Brunn-Minkowski inequality. We show that the method is exact for all convex quadratic polynomials provided the underlying set of nodes is translation invariant within the domain; nodes still conform to the domain boundary. This gives a suitable notion of operator consistency which, combined with stability, leads to pointwise rates of convergence for classical and non-classical solutions of the Monge-Ampère equation.

## 1. INTRODUCTION

We consider the fully nonlinear Monge-Ampère equation

$$\begin{aligned} (1.1a) \quad & \det D^2 u = f && \text{in } \Omega \\ (1.1b) \quad & u = 0 && \text{on } \partial\Omega, \end{aligned}$$

where  $\Omega$  denotes a *uniformly convex* domain in  $\mathbb{R}^d$  ( $d \geq 2$ ) and  $f \in C(\overline{\Omega})$  satisfies  $0 < \lambda_F \leq f(x) \leq \Lambda_F$  for all  $x \in \Omega$ . Such an equation arises from differential geometry, optimal mass transport and several fields of science and engineering, and has received considerable attention in recent years.

In contrast to an extensive PDE literature [30, 10, 17], the numerical approximation is still under development. Convergence to the *viscosity solution* (see Definition 3.3), in the general framework of fully nonlinear elliptic PDEs, is studied in the early works [3, 31] and hinges on operator stability, consistency and monotonicity properties. A. Oberman et al. designed several finite difference methods (wide stencil schemes) within this framework [37, 27, 28, 5]. Benamou et. al. proposed recently a convergent finite difference method, which reduces the stencil size, and showed that the method is consistent for convex polynomials [4].

A geometric notion of generalized solution for the Monge-Ampère equation is the so called *Alexandroff solution* (see Definition 3.2). It hinges on the geometric interpretation of  $\det D^2 u$  as the measure of the subdifferential

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Both authors were partially supported by NSF Grants DMS-1109325 and DMS-1411808.

The second author was partially supported by the Brin Postdoctoral Fellowship of the University of Maryland.

for any convex function  $u$ . This notion of solution is weaker than the viscosity solution. V. Oliker and L. Prussner developed a discrete counterpart and proved convergence of the ensuing geometric numerical method [39]. Even though this is a natural idea, the scheme has defied analysis ever since its conception. The purpose of this paper is to fill this gap upon deriving stability and decay rates in the max norm for the scheme of [39].

Other methods do exist such as the vanishing momentum method of X. Feng and M. Neilan [25, 24, 26], the numerical momentum method of X. Feng et al. [23], the penalty method of S. Brenner et al. [7, 8], nonconforming elements and quadratic elements by M. Neilan [34, 35], standard finite element methods by G. Awanou [2], least squares and augmented Lagrangian methods of E. Dean and R. Glowinski [18, 19, 21, 20, 40], and the  $C^1$  finite element method of K. Böhmer [6]. Error estimates in  $H^1(\Omega)$  are established in [7, 8] for solutions  $u$  with regularity  $H^3$  and above.

The Oliker-Prussner method reads as follows [39]. Let  $\mathcal{N}_h$  be a set of nodes with quasi-uniform spacing  $h$  and let  $\mathcal{T}_h$  be a mesh with nodes  $\mathcal{N}_h$ . The discrete solution  $u_h$  is a *convex nodal function* defined on  $\mathcal{N}_h$ , with vanishing boundary values and satisfying for all interior nodes  $x_i \in \mathcal{N}_h$

$$(1.2) \quad |\partial u_h(x_i)| = f_i,$$

where  $|\cdot|$  denotes the  $d$ -dimensional Lebesgue measure,  $\partial u_h(x_i)$  is the subdifferential of  $u_h$  at  $x_i$  (see (2.10)), and  $f_i = \int_{\Omega} f \phi_i$  is a suitable averaging of  $f$  against the canonical hat basis functions  $\{\phi_i\}$  over  $\mathcal{T}_h$  [39]. Our primary goal in this paper is to establish a bound for the error  $u - u_h$  in the  $L^\infty(\Omega)$ -norm, which seems missing in the current literature for (1.1).

This endeavor entails developing suitable notions of stability, consistency, and monotonicity within the max-norm framework. We now outline the main ingredients. We first need to control the  $L^\infty$ -norm of a function  $v$  by the size of its subdifferential. This is the celebrated *Alexandroff estimate* [30], whose discrete counterpart for a nodal function  $v_h$  is established in [36]: if  $\mathcal{C}_h^-(v_h)$  is the (lower) *contact set* of nodes, namely  $x_i \in \mathcal{N}_h$  so that  $v_h(x_i) = \Gamma(v_h)(x_i)$  with  $\Gamma(v_h)$  the (lower) convex envelope of  $v_h$ , then

$$(1.3) \quad \sup_{\Omega} v_h^- \leq C \left( \sum_{x_i \in \mathcal{C}_h^-(v_h)} |\partial v_h(x_i)| \right)^{1/d},$$

where  $v_h \geq 0$  on  $\partial\Omega$  and  $v_h^-(x) = \max\{-v_h(x), 0\}$  is the negative part of  $v_h$ . Note that the constant  $C = C(d, \Omega)$  is proportional to the diameter of  $\Omega$  and the nodal contact set is just a collection of nodes [36].

Our concept of stability in the max-norm and second ingredient is the following *discrete continuous dependence* estimate, which is a refinement of (1.3) and is derived in Section 4. If  $v_h$  and  $w_h$  are two nodal functions over

$\mathcal{N}_h$  and  $v_h \geq w_h$  at all  $x_i \in \partial\Omega$ , then

$$\sup(v_h - w_h)^- \leq C \left( \sum_{x_i \in \mathcal{C}_h^-(v_h - w_h)} \left( |\partial v_h(x_i)|^{1/d} - |\partial w_h(x_i)|^{1/d} \right)^d \right)^{1/d}.$$

where  $C = C(d, \Omega)$ . The proof of such an estimate relies on a combination of two novel tools from analysis and geometry, namely the discrete Alexandroff estimate (1.3) and the Brunn-Minkowski inequality (see Lemma 4.2). The above estimate will be instrumental to compare the discrete solution  $u_h$  with the nodal function  $N_h u$  associated with the exact solution  $u$ , which is defined as  $N_h u(x_i) = u(x_i)$  for all  $x_i \in \mathcal{N}_h$ .

The third ingredient is *operator consistency*, which is a careful study of the discrepancy between  $u$  and  $N_h u$  carried out in Section 5. We first show that the discrete operator is exact for convex quadratic polynomials  $p$  satisfying  $0 < \lambda I \leq D^2 p \leq \Lambda I$ , namely

$$|\partial N_h p(x_i)| = \int_{\Omega} \phi_i(x) \det D^2 p(x) dx$$

at interior nodes  $x_i \in \mathcal{N}_h$  so that  $\text{dist}(x_i, \partial\Omega) \geq Rh$  with constant  $R = R(\lambda, \Lambda)$ , provided  $\mathcal{N}_h$  is *translation invariant*. For  $u \in C^{2,\alpha}(\overline{\Omega})$ , we immediately deduce that the consistency error is of order  $O(h^\alpha)$  for  $0 < \alpha \leq 1$

$$\left| |\partial N_h u(x_i)| - \int_{\Omega} \phi_i(x) \det D^2 u(x) dx \right| \leq Ch^\alpha |u|_{C^{2,\alpha}(\overline{B_i})} \int_{\Omega} \phi_i(x) dx,$$

where  $B_i := B_{Rh}(x_i)$  is a ball centered at  $x_i$  with radius  $Rh$ . We can also measure the consistency error in Sobolev norms. If  $u \in W_q^s(\Omega)$  with  $\frac{d}{q} + 2 < s \leq 3$ , then we exploit the Sobolev embedding  $W_q^s(\Omega) \subset C^{2,\alpha}(\overline{\Omega})$  with  $\alpha = s - 2 - d/q$  and thus replace  $|u|_{C^{2,\alpha}(\overline{B_i})}$  by  $|u|_{W_q^s(B_i)}$ . Since the set of nodes  $\mathcal{N}_h$  conforms to  $\partial\Omega$ , its translation invariance structure breaks down for nodes close to  $\partial\Omega$ , and so do the consistency bounds which become of order  $O(1)$  and must be handled differently.

Combining the consistency and stability estimates, together with the non-degeneracy assumption  $f \geq \lambda_F > 0$ , we derive the following pointwise convergence rate for  $C^{2,\alpha}$ -solutions in Section 6

$$\|u - \Gamma(u_h)\|_{L^\infty(\Omega)} \leq Ch^\alpha,$$

where the constant  $C = C(d, \Omega, \lambda, \Lambda, \lambda_F) (\|u\|_{C^{2,\alpha}(\overline{\Omega})} + |u|_{W_\infty^2(\Omega)})$ . We point out that the  $C^{2,\alpha}$ -regularity assumption of  $u$  is a consequence of suitable assumptions on  $f$ . In fact, if  $0 < \lambda_F \leq f \leq \Lambda_F$ , then  $0 < \lambda \leq D^2 u \leq \Lambda$  for some constant  $\lambda, \Lambda$ , and if  $f \in C^\alpha(\overline{\Omega})$  and  $\Omega$  is of class  $C^{2,\alpha}$ , then  $u \in C^{2,\alpha}(\overline{\Omega})$  [11], [30, Section 4.3]. We also stress that nodes close to  $\partial\Omega$  are dealt with separately via a discrete barrier argument, which is responsible for the semi-norm  $|u|_{W_\infty^2(\Omega)}$  in the error estimate.

In addition, we prove in Section 6 the following pointwise error estimate for functions with  $W_q^s$ -regularity and  $\frac{d}{q} + 2 < s \leq 3$

$$\|u - \Gamma(u_h)\|_{L^\infty(\Omega)} \leq Ch^{s-2},$$

where the constant  $C = C(d, \Omega, \lambda, \Lambda, \lambda_F)(|u|_{W_q^s(\Omega)} + |u|_{W_\infty^2(\Omega)})$ . This estimate shows that the discrete method (1.2) exhibits first order accuracy in the max-norm provided  $u \in W_q^3(\Omega)$  with  $q > d$ . Since  $u \in W_q^3(\Omega) \subset C^2(\overline{\Omega})$ , the above error estimate is still in the realm of classical solutions. However, our results extend to solutions  $u \in W^{2,\infty}(\overline{\Omega}) \setminus C^2(\overline{\Omega})$  whose Hessian  $D^2u$  is discontinuous across a set  $S$  of (box) dimension  $n < d$ :

$$\|u - \Gamma(u_h)\|_{L^\infty(\Omega)} \leq Ch^{s-2} |u|_{W_q^s(\Omega \setminus S)} + Ch^{\frac{d-n}{d}} |u|_{W_\infty^2(\Omega)}.$$

The rest of this paper is organized as follows. In Section 2 we discuss our notion of discrete convexity, a topic that has received considerable attention recently, as well as we define and explore properties of subdifferentials. In Section 3, we introduce the Alexandroff and viscosity solution concepts for the Monge-Ampère equation (1.1) as well as the geometric numerical method (1.2). We prove stability of (1.2) in Section 4 and consistency in Section 5. We conclude with three rates of convergence depending on solution regularity in Section 6.

## 2. DISCRETE CONVEXITY

Approximating convex solutions of the Monge-Ampère equation (1.1) entails two essential difficulties: dealing with discrete convexity and the fully nonlinear nature of (1.1). The former issue has been investigated in [16, 13, 1, 38, 32], and [16] shows that the space of convex piecewise linear functions on a fixed shape-regular mesh may not be dense in the space of convex functions. In fact, the Lagrange interpolant of a convex function may not even be convex. Several notions of discrete convexity have been proposed in the literature: [13] deals with convex function interpolation on given meshes; [1] introduces finite element functions with positive weak Hessian; [38] imposes positive second order finite differences in all directions within a given stencil. In this paper, we deal with nodal functions and say they are convex if they admit a supporting hyperplane at every node. We make this explicit below.

**2.1. Nodes and Meshes.** In contrast to mesh-based methods, the discretization of (1.1) hinges on a collection of nodes  $\{x_i\}_{i=1}^n$  within the domain  $\Omega$  and  $\{x_i\}_{i=n+1}^N$  on  $\partial\Omega$ , and a collection of simplices (or elements)  $T$  with nodes  $x_i$  which form a mesh of  $\Omega$ . For each element  $T$ , we denote by  $h_T$  the diameter of  $T$  and by  $\rho_T$  the diameter of the largest inscribed ball in  $T$ . For each node  $x_i$ , we define the local spacing at  $x_i$  as

$$h_i := \max\{h_T : x_i \in T\}.$$

We say that the nodal set is *quasi-uniform* if there exist constants  $0 < \gamma \leq 1$  and  $h > 0$  such that  $\gamma h \leq h_i \leq h$  for all  $1 \leq i \leq n$ . We define the shape-regularity constant of an element  $T$  to be

$$\sigma_T := \frac{h_T}{\rho_T}$$

and we say that the nodal set is *shape-regular* if there exists  $\sigma > 0$  such that  $\sigma_T \leq \sigma$  for all elements  $T$ . We will assume throughout that the nodal set is quasi-uniform and shape-regular and write

$$\mathcal{N}_h^0 := \{x_i\}_{i=1}^n, \quad \mathcal{N}_h^\partial := \{x_i\}_{i=n+1}^N, \quad \mathcal{N}_h := \mathcal{N}_h^0 \cup \mathcal{N}_h^\partial,$$

as well as  $\mathcal{T}_h^0 := \{T\}$  for the original mesh; other meshes  $\mathcal{T}_h$  will appear later. We say that  $\mathcal{N}_h$  is *translation invariant* if there is a basis  $\{e_j\}_{j=1}^d$  of  $\mathbb{R}^d$  with  $|e_j| \leq 1$  for all  $1 \leq j \leq d$  so that the interior nodes satisfy

$$(2.1) \quad \mathcal{N}_h^0 = \left\{ x_i = h \sum_{j=1}^d k_j e_j : k_j \in \mathbb{Z} \right\} \cap \Omega,$$

whereas the boundary nodes are only required to have nodal spacing  $h$

$$\partial\Omega \subset \cup_{x_i \in \mathcal{N}_h^\partial} B_{h/2}(x_i).$$

Obviously, a cartesian lattice with spacing  $\sqrt{d}h$  is translation invariant and  $\{e_j\}_{j=1}^d$  are the canonical unit vectors in  $\mathbb{R}^d$ .

Given a mesh  $\mathcal{T}_h$  with nodes  $\mathcal{N}_h$ , we denote by  $\{\phi_i\}_{i=1}^n$  the canonical basis of piecewise linear functions associated with  $\mathcal{N}_h^0$ . We say that  $\{\phi_i\}_{i=1}^n$  is *translation invariant* provided that for all  $x_i, x_j \in \mathcal{N}_h^0$  such that  $\text{dist}(x_i, \partial\Omega), \text{dist}(x_j, \partial\Omega) > h$  and for all  $x \in \mathbb{R}^d$  we have

$$(2.2) \quad \phi_i(x + x_i) = \phi_j(x + x_j).$$

**2.2. Convexity of Nodal Functions.** We say that the nodal function  $u_h : \mathcal{N}_h \rightarrow \mathbb{R}$  is *convex* if for all  $x_i \in \mathcal{N}_h^0$  there is a supporting hyperplane  $L$  of  $u_h$ , that is

$$L(x_j) \leq u_h(x_j) \text{ for all } x_j \in \mathcal{N}_h \text{ and } L(x_i) = u_h(x_i).$$

We define the convex envelope of a nodal function  $u_h$  to be

$$(2.3) \quad \Gamma(u_h)(x) = \sup_L \{L(x) \text{ affine} : L(x_i) \leq u_h(x_i) \quad \forall x_i \in \mathcal{N}\} \quad \forall x \in \Omega.$$

If the nodal function  $u_h$  is convex, then we have

$$(2.4) \quad u_h(x_i) = \Gamma(u_h)(x_i) \quad \text{for all } x_i \in \mathcal{N}_h^0.$$

We regard  $\Gamma(u_h)$  as a natural convex extension of  $u_h$  and call it the *convex interpolant* of  $u_h$ . On the other hand, given a continuous function  $u : \overline{\Omega} \rightarrow \mathbb{R}$  we denote by  $N_h u : \mathcal{N}_h \rightarrow \mathbb{R}$  the nodal function associated with  $u$ :

$$(2.5) \quad N_h u(x_i) = u(x_i) \quad \forall x_i \in \mathcal{N}_h.$$

Since  $\Gamma(u_h)(x)$  is a piecewise linear function, it induces a mesh  $\mathcal{T}_h$  which depends on  $u_h$ . Given a convex nodal function  $u_h$  and a node  $x_i \in \mathcal{N}_h^0$ , we define the set  $A_i(u_h)$  of *adjacent nodes* (or *adjacent set*) of  $x_i$  for  $u_h$  as a collection of nodes  $x_j \in \mathcal{N}_h$  such that there exists a supporting hyperplane  $L$  of  $u_h$  at  $x_i$  and  $L(x_j) = u_h(x_j)$ . The set  $A_i(u_h)$  is the collection of nodes in the star associated with  $x_i$  in the mesh  $\mathcal{T}_h$  induced by  $\Gamma(u_h)$ . The following example illustrates that such a star could be quite elongated even for a cartesian lattice.

**Example 2.1** (anisotropic star). Let  $\Omega = \mathbb{R}^2$ ,  $\mathcal{N}_h = \mathbb{Z}^2$  and  $h = 1$ . Let  $u(x) = (x \cdot e)^2$  where  $e = (1, m)$  for some integer  $m \geq 1$  and  $u_h(x_i) = u(x_i)$  for all  $x_i \in \mathcal{N}_h$ . Then the convex envelope  $\Gamma(u_h)$  of  $u_h$  induces an anisotropic mesh  $\mathcal{T}_h$ ; Figure 2.1 displays the star associated to the origin for  $m = 1$ . The convex envelope  $\Gamma(u_h)$  in such a star is  $\Gamma(u_h)(x) = |x \cdot e|$ .

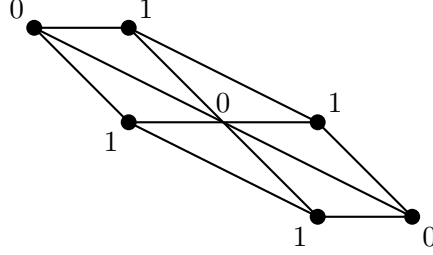


FIGURE 2.1. Anisotropic star at  $(0,0)$  induced by the convex envelope  $\Gamma(u_h)$  of the nodal function  $u_h(x_i) = (x_i \cdot e)^2$  where  $e = (1, 2)$ . Note that  $\Gamma(u_h)(x) = |x \cdot e|$  in the star.

This example shows that if the Hessian  $D^2u$  is degenerate or nearly degenerate, then the mesh  $\mathcal{T}_h$  induced by  $\Gamma(u_h)$  may be quite anisotropic.

Given a mesh  $\mathcal{T}_h$  with nodes  $\mathcal{N}_h$  we denote by  $I_h(u_h)$  the Lagrange interpolant  $I_h(u_h)$  of  $u_h$  over  $\mathcal{T}_h$ , namely the continuous piecewise linear function that interpolates the nodal values of  $u_h$  over  $\mathcal{T}_h$ . The following property is helpful to check discrete convexity: given a mesh  $\mathcal{T}_h$  with nodes  $\mathcal{N}_h$  and a nodal function  $u_h$ ,  $I_h(u_h)$  is convex if and only if it satisfies [36, Lemma 5.3]

$$(2.6) \quad \llbracket \nabla I_h(u_h) \rrbracket_F \geq 0 \text{ for all faces } F$$

where  $F = K^+ \cap K^-$  with  $K^\pm \in \mathcal{T}_h$ , the jump is given by

$$(2.7) \quad \llbracket \nabla I_h(u_h) \rrbracket_F := -n_F^+ \cdot \nabla I_h(u_h)|_{K^+} - n_F^- \cdot \nabla I_h(u_h)|_{K^-}$$

and  $n_F^\pm$  are the outer normal vectors of  $K_\pm$  on face  $F$ . If  $I_h(u_h)$  is convex, then  $I_h(u_h) = \Gamma(u_h)$ .

We finally let the (nodal lower) *contact set* of a nodal function  $u_h$  be

$$(2.8) \quad \mathcal{C}_h^-(u_h) := \{x_i \in \mathcal{N}_h^0 : \Gamma(u_h)(x_i) = u_h(x_i)\}.$$

Note that if  $u_h$  is convex, then  $\mathcal{C}_h^-(u_h) = \mathcal{N}_h^0$ ; otherwise,  $\mathcal{C}_h^-(u_h) \subset \mathcal{N}_h^0$ .

**2.3. Subdifferential.** Let  $u : \Omega \rightarrow \mathbb{R}$  be a convex function and  $x_0 \in \Omega$ . The subdifferential of  $u$  at  $x_0$  is the set

$$(2.9) \quad \partial u(x_0) = \{v \in \mathbb{R}^d : u(x) \geq u(x_0) + v \cdot (x - x_0)\}.$$

Given a set  $S \subset \Omega$ , we define

$$\partial u(S) = \cup_{x \in S} \partial u(x).$$

Since  $u$  is a convex function, the subdifferential  $\partial u(x)$  is non-empty and convex for all  $x \in \Omega$ .

Similarly, we define the subdifferential of nodal function  $u_h$  at node  $x_i$  by

$$(2.10) \quad \partial u_h(x_i) = \{v \in \mathbb{R}^d : u_h(x_j) \geq u_h(x_i) + v \cdot (x_j - x_i) \quad \forall x_j \in \mathcal{N}_h\}.$$

If the nodal function  $u_h$  is convex, then  $\partial u_h(x_i)$  is non-empty for all  $x_i \in \mathcal{N}_h^0$ . The following lemma relates definitions (2.9) and (2.10).

**Lemma 2.1** (discrete subdifferential). *If  $u_h$  is a convex nodal function, then  $\partial u_h(x_i) = \partial \Gamma(u_h)(x_i)$  for all  $x_i \in \mathcal{N}_h^0$ .*

*Proof.* Obviously, if  $v \in \partial \Gamma(u_h)(x_i)$ , that is,

$$\Gamma(u_h)(x) \geq \Gamma(u_h)(x_i) + v \cdot (x - x_i) \quad \forall x \in \Omega,$$

then  $v \in \partial u_h(x_i)$  thanks to (2.4).

Conversely, let  $\mathcal{T}_h$  be a mesh induced by the convex envelope  $\Gamma(u_h)$ , let  $K \in \mathcal{T}_h$  be an element with vertices  $\{x_j\}$ . If  $v \in \partial u_h(x_i)$ , then  $u_h(x_j) \geq u_h(x_i) + v \cdot (x_j - x_i)$  for all  $x_j \in \mathcal{N}_h$ , whence again thanks to (2.4), we have  $\Gamma(u_h)(x_j) \geq \Gamma(u_h)(x_i) + v \cdot (x_j - x_i)$  for all vertices  $x_j$  of  $K$ . Since  $\Gamma(u_h)$  is linear in  $K$ , we have  $\Gamma(u_h)(x) \geq \Gamma(u_h)(x_i) + v \cdot (x - x_i)$  for any  $x \in K$ . This shows that  $v \in \partial \Gamma(u_h)(x_i)$  as well.  $\square$

**Lemma 2.2** (subdifferential monotonicity). *Let  $u_h$  and  $v_h$  be two convex nodal functions. If  $u_h(x_i) \leq v_h(x_i)$  and  $u_h(x_j) = v_h(x_j)$  for all  $j \neq i$ , then*

$$\partial v_h(x_i) \subset \partial u_h(x_i) \quad \text{and} \quad \partial u_h(x_j) \subset \partial v_h(x_j) \quad \text{for all } j \neq i.$$

*Proof.* This follows directly from the definition of subdifferential (2.10).  $\square$

Given two compact sets  $A, B$  of  $\mathbb{R}^d$ , their *Minkowski sum* is given by

$$(2.11) \quad A + B := \{v + w \in \mathbb{R}^d : v \in A \text{ and } w \in B\}.$$

**Lemma 2.3** (addition of subdifferentials). *If  $u_h$  and  $v_h$  be two convex nodal functions, then*

$$\partial w_h(x_i) + \partial v_h(x_i) \subset \partial (w_h + v_h)(x_i) \quad \text{for all } x_i \in \mathcal{N}_h^0.$$

*Proof.* We note that if  $w \in \partial w_h(x_i)$  and  $v \in \partial v_h(x_i)$ , then

$$w_h(x_j) \geq w_h(x_i) + w \cdot (x_j - x_i) \quad \text{and} \quad v_h(x_j) \geq v_h(x_i) + v \cdot (x_j - x_i)$$

for all  $x_j \in \mathcal{N}_h$ . Adding both inequalities yields

$$w_h(x_j) + v_h(x_j) \geq w_h(x_i) + v_h(x_i) + (w + v) \cdot (x_j - x_i)$$

which implies  $w + v \in \partial (w_h + v_h)(x_i)$ .  $\square$

Computing the subdifferential  $\partial u_h(x_i)$  of a given convex nodal function  $u_h$  at  $x_i \in \mathcal{N}_h^0$  is a nontrivial task because it is nonlocal. In fact, Lemma 2.1 shows that it involves computing the convex envelope  $\Gamma(u_h)$  of  $u_h$ . The following lemma, proved in [36, Lemma 5.4], characterizes  $\partial\Gamma(u_h)$ .

**Lemma 2.4** (characterization of subdifferential). *Let  $u_h$  be a convex nodal function and  $\mathcal{T}_h$  be the mesh induced by its convex envelope  $\Gamma(u_h)$ . Then the subdifferential of  $u_h$  at  $x_i \in \mathcal{N}_h^0$  is the convex hull of the constant gradients  $\nabla\Gamma(u_h)|_K$  for all  $K \in \mathcal{T}_h$  which contain  $x_i$ .*

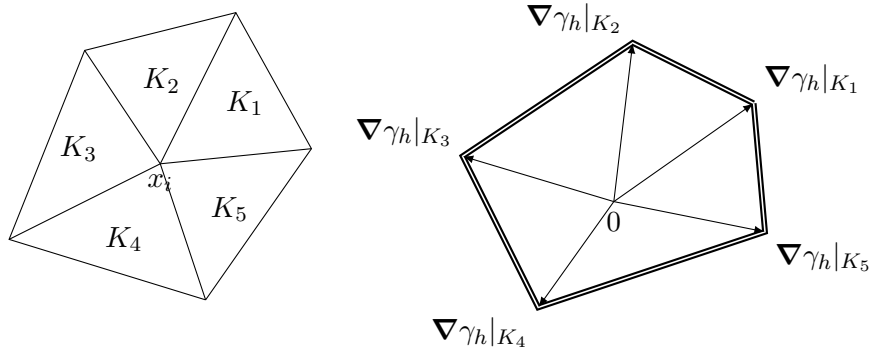


FIGURE 2.2. Star centered at node  $x_i$  corresponding to the mesh  $\mathcal{T}_h$  induced by the convex envelope  $\gamma_h = \Gamma(u_h)$  of  $u_h$  and subdifferential  $\partial u_h(x_i)$  of the convex nodal function  $u_h$  at node  $x_i$  such that  $0 \in \partial u_h(x_i)$ . The latter is the convex hull of the constant element gradients  $\nabla\gamma_h|_{K_j}$  for  $1 \leq j \leq 5$ .

Figure 2.2 depicts the subdifferential  $\partial u_h(x_i)$  of the convex nodal function  $u_h$  at node  $x_i$  for  $d = 2$ . Computing  $\partial u_h(x_i)$  is equivalent to finding a mesh  $\mathcal{T}_h$  such that the jumps  $\llbracket \nabla I_h(u_h) \rrbracket_F$  of the Lagrange interpolant of  $I_h(u_h)$  on faces  $F$  are nonnegative.

Unfortunately, the notion of discrete convexity is not sufficiently robust geometrically. Consider a pair of elements  $K^\pm \in \mathcal{T}_h$  with common face  $F = K^+ \cap K^-$ , and assume  $\llbracket \nabla I_h(u_h) \rrbracket_F = 0$ . Increasing the values of  $u_h$  at the nodes opposite to  $F$  increases the jump, and thus preserves convexity of  $I_h(u_h)$  on  $\mathcal{T}_h$ , whereas decreasing the values of  $u_h$  violates convexity on  $\mathcal{T}_h$ .

**Lemma 2.5** (geometric stability). *Let  $u_h$  be a nodal function. If the convex envelope  $\Gamma(u_h)$  of  $u_h$  has positive jumps on all faces of the induced mesh  $\mathcal{T}_h$ , then  $\mathcal{T}_h$  is invariant under small nodal perturbations of  $u_h$ .*

*Proof.* Since the jumps  $\llbracket \nabla\Gamma(u_h) \rrbracket_F$  are continuous with respect to nodal variations of  $u_h$ , and they are positive, they remain positive under small nodal perturbations. We then apply (2.6) to deduce the assertion.  $\square$



### 3. SOLUTION AND APPROXIMATION

There are two notions of weak solutions of (1.1): the Alexandroff solution hinges on a geometric interpretation of (1.1) and the viscosity solution relies on the comparison principle. We review these definitions now and discuss a geometric approximation of (1.1) due to Oliker and Prussner [39].

**3.1. Alexandroff Solution.** To motivate this solution concept, suppose for the moment that  $u \in C^2(\Omega)$  is a strictly convex function satisfying (1.1) in a classical sense. The convexity and  $C^2$ -regularity assumptions imply that  $\partial u(x) = \nabla u(x)$  and that the subdifferential, viewed as a map  $\partial u : \Omega \rightarrow \mathbb{R}^d$ , is injective. Consequently, the change of variables  $y = \nabla u(x)$  reveals that

$$\int_D f \, dx = \int_D \det D^2 u(x) \, dx = \int_{\partial u(D)} dy = |\partial u(D)|$$

for all Borel sets  $D \subset \Omega$ , where  $\partial u(D) = \cup_{x \in D} \partial u(x)$  and  $|\cdot|$  is the  $d$ -dimensional Lebesgue measure. Since  $\partial u$  is well-defined for non-smooth convex functions, the above identity allows one to widen the class of admissible solutions [30, Section 1.2].

**Definition 3.1** (Monge-Ampère measure). *We define the Monge-Ampère measure associated with a convex function  $u \in C(\Omega)$  as*

$$Mu(S) = |\partial u(S)|$$

where  $|\cdot|$  denotes the  $d$ -dimensional Lebesgue measure.

**Definition 3.2** (Alexandroff solution). *Let  $\mu$  be a Borel measure defined in  $\Omega$ , an open and convex subset of  $\mathbb{R}^d$ . We say a convex function  $u \in C(\Omega)$  is an Alexandroff solution to the Monge-Ampère equation*

$$\det D^2 u = \mu$$

*if the Monge-Ampère measure  $Mu$  associated with  $u$  equals  $\mu$ .*

The Alexandroff solution is closed under uniform convergence. This is stated in the lemma below and its proof is given in [30, Lemma 1.2.3].

**Lemma 3.1** (weak convergence of Monge-Ampère measures). *If  $u_k$  are convex functions in  $\Omega$  such that  $u_k \rightarrow u$  as  $k \rightarrow \infty$  uniformly on compact subsets of  $\Omega$ , then the associated Monge-Ampère measures  $Mu_k$  tend to  $Mu$  weakly, that is*

$$\int_{\Omega} \phi(x) dMu_k(x) \rightarrow \int_{\Omega} \phi(x) dMu(x) \quad \text{as } k \rightarrow \infty,$$

*for every  $\phi$  continuous with compact support in  $\Omega$ .*

**3.2. Viscosity Solution.** This concept hinges on the comparison principle.

**Definition 3.3** (viscosity solution). *Let  $u \in C(\Omega)$  be a convex function and  $f \in C(\Omega)$ ,  $f > 0$ . The function  $u$  is a viscosity sub-solution (super-solution) of the Monge-Ampère equation in  $\Omega$  if whenever a convex function  $\phi \in C^2(\Omega)$  and  $x_0 \in \Omega$  are such that  $(u - \phi)(x) \leq (\geq)(u - \phi)(x_0)$  for all  $x$  in a neighborhood of  $x_0$ , then we must have*

$$\det D^2\phi(x_0) \geq (\leq)f(x_0).$$

If  $u \in C(\Omega)$  is an Alexandroff solution with  $Mu = f$  and  $f \in C(\Omega)$ , then  $u$  is a viscosity solution [30, Proposition 1.3.4]. Conversely, if  $f \in C(\overline{\Omega})$  and  $f > 0$ , then the viscosity solution is also the Alexandroff solution [30, Proposition 1.7.1].

**3.3. Examples of Weak Solutions.** We show examples of Alexandroff and viscosity solutions, which are not classical solutions, namely  $u \notin C^2(\overline{\Omega})$ .

**Example 3.1** (Alexandroff solution). Let  $\Omega = B_1(0) \subset \mathbb{R}^2$  and

$$u(x) = |x| - 1.$$

The function  $u(x)$  is an Alexandroff solution of Monge-Ampère equation

$$\det D^2u = \pi\delta_{(x=0)},$$

where  $\delta_{(x=0)}$  is the Dirac measure at origin, but  $u$  is not a viscosity solution.

**Example 3.2** (viscosity solution). Let  $\Omega = B_2(0) \subset \mathbb{R}^2$  and [27]

$$(3.1) \quad u(x) = \begin{cases} 0 & \text{in } |x| \leq 1, \\ \frac{1}{2}(|x| - 1)^2 & \text{in } 1 \leq |x| \leq 2. \end{cases}$$

The function  $u$  is a viscosity solution of Monge-Ampère equation

$$\det D^2u(x) = \begin{cases} 0 & \text{in } |x| \leq 1, \\ 1 - |x|^{-1} & \text{in } 1 \leq |x| \leq 2. \end{cases}$$

We note that  $u \in C^{1,1}(\overline{\Omega}) \setminus C^2(\overline{\Omega})$ , and that  $f = 0$  in the unit ball  $B_1(0)$ , which means that (1.1) is degenerate.

**3.4. Oliker-Prussner Method.** Following [39] we can approximate (1.1) exploiting its geometric interpretation. Given a quasi-uniform nodal set  $\mathcal{N}_h$  of  $\Omega$  and a function  $f > 0$ , we start by defining a nodal function  $f_h : \mathcal{N}_h \rightarrow \mathbb{R}$  consistent with  $f$ . In [39], the nodal values  $f_i = f_h(x_i)$  are taken to be

$$f_i = \int_{\omega_i} f(x) dx \quad \forall x_i \in \mathcal{N}_h^0$$

where  $\{\omega_i\}_{x_i \in \mathcal{N}_h^0}$  is a partition of the domain  $\Omega$  into  $n$  disjoint sub-domains  $\omega_i$  containing  $x_i$ . Since implementing such a partition and subsequent computation of integrals may not be an easy task, we propose the following

variant. Let the original mesh  $\mathcal{T}_h^0$  of  $\Omega$  be a Delaunay triangulation with nodal set  $\mathcal{N}_h$ , and set

$$(3.2) \quad f_i := \int_{\Omega} f(x) \phi_i(x) dx \quad \forall x_i \in \mathcal{N}_h^0$$

where  $\phi_i$  is the linear hat basis function associated with node  $x_i$  over  $\mathcal{T}_h^0$ .

The discretization of (1.1) reads as follows: we seek a convex nodal function  $u_h$  satisfying

$$(3.3) \quad |\partial u_h(x_i)| = f_i \quad \forall x_i \in \mathcal{N}_h^0.$$

We refer to [39] for a proof of existence and uniqueness of (3.3). Below we give a proof of existence that ties the concepts developed so far together.

**3.5. Existence: Discrete Perron's Method.** We assume that the set of nodes  $\mathcal{N}_h$  is shape-regular and construct a sequence  $\{u_h^k\}_{k=1}^{\infty}$  of convex nodal functions which is monotone, namely  $u_h^{k+1}(x_i) \geq u_h^k(x_i)$  for all  $x_i \in \mathcal{N}_h^0$ , and converges to a solution of (1.1). The purpose of this section is to illustrate how the corresponding meshes  $\mathcal{T}_h^k$  change with the iteration parameter  $k$ .

**Construction of  $u_h$ .** We first initialize the iteration. We assume that  $\Omega$  is contained in a ball  $B_R(0)$  of radius  $R$  centered at the origin and  $d = 2$ . We consider the quadratic polynomial  $p(x) := \frac{\Lambda^{1/2}}{2} (|x|^2 - R^2)$  with  $\Lambda > 0$  to be specified later. Then

$$\det D^2 p(x) = \Lambda \text{ for any } x \in \Omega \quad \text{and} \quad p(x) \leq 0 \text{ on } \partial\Omega.$$

We define  $u_h^0 = N_h p$ , namely

$$u_h^0(x_i) := p(x_i) \quad \forall x_i \in \mathcal{N}_h.$$

Since  $\mathcal{T}_h^0$  is a Delaunay triangulation associated with  $\mathcal{N}_h$ , the Lagrange interpolant of  $u_h^0$  is convex [14, 15, 22], and thus coincides with the convex envelope of  $u_h^0$ :

$$I_h(u_h^0) = \Gamma(u_h^0).$$

We assert that  $u_h^0$  is a subsolution of (3.3). To see this we need to estimate the measure of the subdifferential  $|\partial u_h^0(x_i)|$  at every node  $x_i \in \mathcal{N}_h^0$ . Since  $\mathcal{N}_h^0$  is shape-regular, the star  $\omega_i$  of  $\mathcal{T}_h^0$  around  $x_i$  has a diameter proportional to  $h$  and contains a ball  $B_{\alpha h}(x_i)$  of radius  $\alpha h$  with  $\alpha > 0$  only dependent on the shape regularity constant  $\sigma$ . Upon subtracting the affine function  $L(x) = p(x_i) + \nabla p(x_i) \cdot (x - x_i)$  from  $p$ , we can assume that  $p$  and its gradient vanish at  $x = x_i$  without changing  $|\partial u_h^0(x_i)|$ . Since  $p(x) = \frac{\Lambda^{1/2}}{2} |x - x_i|^2$ , we see that  $p(x) \geq \frac{\Lambda^{1/2}}{2} \alpha^2 h^2$  on  $\partial\omega_i$  and we can apply the discrete Alexandroff estimate (1.3) to  $u_h^0 - \frac{\Lambda^{1/2}}{2} \alpha^2 h^2$  on  $\omega_i$  to deduce

$$|\partial u_h^0(x_i)|^{1/2} \geq C h^{-1} \sup_{\omega_i} \left( u_h^0 - \frac{\Lambda^{1/2}}{2} \alpha^2 h^2 \right)^- \geq C \Lambda^{1/2} h,$$

whence

$$|\partial u_h^0(x_i)| \geq C \Lambda h^2 = C \Lambda |\omega_i|.$$

Recalling that  $f_i = \int_{\omega_i} f \phi_i \leq \Lambda_F |\omega_i|$ , we can choose  $\Lambda > 0$  sufficiently large so that

$$(3.4) \quad |\partial u_h^0(x_i)| \geq f_i \quad \forall x_i \in \mathcal{N}_h^0.$$

Moreover, in view of  $u_h^0 \leq 0$  on  $\partial\Omega$  we infer that  $u_h^0$  is a subsolution of (3.3), as asserted.

To construct the iterates  $u_h^{k+1}$  for  $k \geq 0$  we recall Lemma 2.2 (subdifferential monotonicity): if the nodal value  $u_h^k(x_i)$  increases, then  $|\partial u_h^k(x_i)|$  decreases and  $|\partial u_h^k(x_j)|$  increases for all other nodes  $x_j$  ( $j \neq i$ ). With this in mind, we first set the boundary values one at the time

$$u_h^1(x_i) = 0 \quad \forall x_i \in \mathcal{N}_h^\partial,$$

which preserves (3.4) and the convexity of  $u_h^1$ . If  $u_h^k$  convex has been computed, we now describe how to construct  $u_h^{k+1}$  upon increasing the internal nodal values  $u_h^k(x_i)$  one at the time until

$$|\partial u_h^{k+1}(x_i)| = f_i \quad x_i \in \mathcal{N}_h^0,$$

which implies for  $j \neq i$

$$|\partial u_h^{k+1}(x_j)| \geq f_j \quad x_j \in \mathcal{N}_h^0.$$

This is always doable because  $|\partial u_h^{k+1}(x_i)|$  decreases continuously with increasing  $u_h^{k+1}(x_i)$  until it vanishes, which corresponds to having a supporting hyperplane at  $x_i$  that touches  $u_h^{k+1}$  at  $d+1$  nodes distinct from  $x_i$  and not lying in one hyperplane. In addition, since  $f_i > 0$  for all  $x_i \in \mathcal{N}_h^0$ , the intermediate iterates leading to  $u_h^{k+1}$  are always convex by definition. This process creates a monotone sequence  $\{u_h^k\}_{k \in \mathbb{N}}$  of convex nodal functions, namely  $u_h^{k+1}(x_i) \geq u_h^k(x_i)$  for all  $x_i \in \mathcal{N}_h^0$ . Since this sequence has a uniform upper bound, namely  $u_h^k(x_i) \leq 0$  for all  $x_i \in \mathcal{N}_h^0$  and all  $k$ , as a consequence of Corollary 4.4 (maximum principle) below, we deduce that the sequence converges to a nodal function  $u_h$ . Next, we show that the limit  $u_h$  satisfies (3.3). Taking  $\phi(x)$  in Lemma 3.1 as the hat function  $\phi_i$  associated with  $x_i \in \mathcal{N}_h^0$ , we deduce that  $|\partial u_h^k(x_i)|$  converges and

$$\begin{aligned} |\partial u_h(x_i)| &= \int_{\Omega} \phi_i(x) dMu_h(x) \\ &= \lim_{k \rightarrow \infty} \int_{\Omega} \phi_i(x) dMu_h^k(x) = \lim_{k \rightarrow \infty} |\partial u_h^k(x_i)| \geq f_i. \end{aligned}$$

We argue by contradiction. If  $u_h$  does not satisfy (3.3), then there exists a node  $x_i \in \mathcal{N}_h^0$  such that  $|\partial u_h(x_i)| > f_i$  and there exists  $\delta > 0$  such that increasing  $u_h(x_i)$  by  $\delta$ , the subdifferential at  $x_i$  equals  $f_i$ . On the other hand, given any  $\epsilon > 0$ , there exists  $k_\epsilon$  such that  $0 \leq u_h(x_j) - u_h^k(x_j) \leq \epsilon$  for all  $k \geq k_\epsilon$ ,  $x_j \in \mathcal{N}_h^0$ . We define the auxiliary function  $\tilde{u}_h^k(x_j) := u_h^{k+1}(x_j)$  if  $j < i$ ,  $\tilde{u}_h^k(x_j) := u_h^k(x_j)$  if  $j > i$  and  $\tilde{u}_h^k(x_i) := u_h^k(x_i) + \delta - \epsilon$ . We note that

$$\tilde{u}_h^k(x_j) + \epsilon \geq u_h(x_j) \quad j \neq i, \quad \tilde{u}_h^k(x_i) + \epsilon = u_h^k(x_i) + \delta \leq u_h(x_i) + \delta,$$

whence, applying Lemma 2.2 to  $\tilde{u}_h^k + \epsilon$  and  $u_h$  perturbed by  $\delta$  at  $x_i$  yields

$$|\partial \tilde{u}_h^k(x_i)| \geq f_i.$$

Therefore, since  $u_h^{k+1}(x_i)$  is the largest value satisfying  $|\partial u_h^{k+1}(x_i)| = f_i$  we deduce  $u_h^{k+1}(x_i) \geq \tilde{u}_h^k(x_i) \geq u_h(x_i) + \delta - 2\epsilon > u_h(x_i)$  provided that  $\epsilon < \delta/2$ . This contradicts the fact that  $u_h^{k+1}(x_i) \leq u_h(x_i)$  and proves (3.3).

**Computation of subdifferentials.** Computing  $|\partial u_h^k(x_i)|$  is a key step of the algorithm, which reduces to computing the constant gradients  $\nabla \Gamma(u_h^k)|_K$  of the convex envelope  $\Gamma(u_h^k)$  of  $u_h^k$  for each element  $K$  of the induced mesh  $\mathcal{T}_h^k$ . Lemma 2.4 yields

$$|\partial u_h^k(x_i)| = \text{measure of polygon with vertices } \{\nabla \Gamma(u_h^k)|_K\} \text{ and } x_i \in K.$$

During the iteration the underlying mesh  $\mathcal{T}_h^k$  changes, starting from the Delaunay mesh for  $\mathcal{N}_h$ . We describe now how these changes occur and can be implemented.

We consider a simple example in  $\mathbb{R}^2$ . Let  $\mathcal{T}_h^k$  consist of two triangles  $K_1, K_2$  with vertices  $z_1, z_2, z_4$  and  $z_2, z_3, z_4$ , namely

$$K_1 = \text{convex hull}(z_1, z_2, z_4), \quad K_2 = \text{convex hull}(z_2, z_3, z_4),$$

and

$$z_1 = (h, 0), \quad z_2 = (0, -h), \quad z_3 = (-h, 0), \quad z_4 = (0, h).$$

For  $t \in [-1, 1]$ , we consider the one-parameter convex nodal function

$$u_h^k(z_i) = 0 \quad i = 1, 2, 3, \quad u_h^k(z_4) = t,$$

and observe that its Lagrange interpolant  $I_h(u_h^k)$  is convex for  $-1 \leq t \leq 0$  and concave for  $0 \leq t \leq 1$ ; hence  $\Gamma(u_h^k) = I_h(u_h^k)$  only for  $-1 \leq t \leq 0$ . The constant gradients in  $K_i$  are  $t \nabla \phi_4$  for  $i = 1, 2$  and the jump on the edge  $F_1 = [z_2, z_4]$  is given by

$$\llbracket \nabla I_h(u_h^k) \rrbracket|_{F_1} = -\frac{t}{h}.$$

We see that it is non-negative only for  $-1 \leq t \leq 0$ , which means that  $I_h(u_h^k)$  is convex according to (2.6). When  $t = 0$  the function  $I_h(u_h^k)$  is linear in  $K_1 \cup K_2$  and for  $0 \leq t \leq 1$  we have to flip the edge  $F_1$  to  $F_2 = [z_1, z_3]$  and consider a new mesh made of the two new triangles

$$K_3 = \text{convex hull}(z_1, z_2, z_3), \quad K_4 = \text{convex hull}(z_1, z_3, z_4),$$

for which  $\llbracket \nabla I_h(u_h^k) \rrbracket|_{F_2} = \frac{t}{h} \geq 0$  and  $I_h(u_h^k)$  is convex. This example reveals that the connectivity of the underlying mesh  $\mathcal{T}_h^k$  may change as we increase nodal values  $u_h^k(x_i)$ , but the change is local, say within the star  $\omega_i$ , and we can monitor it by simply checking the signs of jumps within  $\omega_i$  and flipping edges accordingly. Since increasing  $u_h^k(x_i)$  is equivalent to adding a multiple of  $\phi_i$  to  $u_h^k$ , we realize that the jumps on the boundary edges of  $\omega_i$  increase, which is consistent with Lemma 2.2, and require no attention whatsoever.

In the rest of the paper, we focus on estimating the rates of convergence. Such an estimate relies on the stability and consistency of (3.3), which we address in sections 4 and 5, respectively.

#### 4. STABILITY

Given two nodal functions  $v_h$  and  $w_h$  we control the  $L^\infty$ -norm of  $(v_h - w_h)^-$  in terms of the discrepancy of their subdifferential measures. This is the content of Proposition 4.3 and its proof is the chief goal of this section. This result hinges on two important estimates, the discrete Alexandroff estimate and the Brunn-Minkowski inequality, which we discuss next.

**4.1. Discrete Alexandroff Estimate.** The Alexandroff estimate for a continuous piecewise affine function  $v_h$  states that the  $L^\infty$ -norm of  $v_h$  is controlled by the Lebesgue measure of its subdifferential. We refer to [36] for a complete proof.

**Lemma 4.1** (discrete Alexandroff estimate). *Let  $v_h$  be a nodal function and  $v_h(x_i) \geq 0$  at all  $x_j \in \partial\Omega$ . Then*

$$(4.1) \quad \sup_{\Omega} v_h^- \leq C \left( \sum_{x_i \in \mathcal{C}_h^-(v_h)} |\partial v_h(x_i)| \right)^{1/d},$$

where  $C = C(d, \Omega)$  is proportional to the diameter of  $\Omega$  and  $\mathcal{C}_h^-(v_h)$  is the contact set defined in (2.8).

**4.2. Brunn-Minkowski Inequality.** The second main tool we use to prove Lemma 4.3 is the celebrated Brunn-Minkowski inequality [29]. This inequality relates the Lebesgue measures of compact subsets  $A, B$  of Euclidean space  $\mathbb{R}^d$  with that of their Minkowski sum  $A + B$  defined in (2.11).

**Lemma 4.2** (Brunn-Minkowski inequality). *Let  $A$  and  $B$  be two nonempty compact subsets of  $\mathbb{R}^d$  for  $d \geq 1$ . Then the following inequality holds:*

$$|A + B|^{1/d} \geq |A|^{1/d} + |B|^{1/d}.$$

Since  $(a + b)^t \leq a^t + b^t$  for  $a, b \geq 0$  and  $0 < t < 1$ , we deduce the following immediate consequence of Lemma 4.2

$$(4.2) \quad |A + B| \geq |A| + |B|.$$

**4.3. Discrete Continuous Dependence.** We now compare two arbitrary nodal functions in terms of their subdifferentials. This is instrumental for the error analysis.

**Proposition 4.3** (stability of numerical solution). *Let  $v_h$  and  $w_h$  be two nodal functions associated with nodes  $\mathcal{N}_h$  and  $v_h \geq w_h$  at all  $x_i \in \partial\Omega$ . Then*

$$\sup(v_h - w_h)^- \leq C \left( \sum_{x_i \in \mathcal{C}_h^-(v_h - w_h)} \left( |\partial v_h(x_i)|^{1/d} - |\partial w_h(x_i)|^{1/d} \right)^d \right)^{1/d},$$

where  $C = C(d, \Omega)$  is proportional to the diameter of  $\Omega$ .

*Proof.* Let  $v_h, w_h$  be two arbitrary nodal functions. We consider the convex envelope  $\Gamma(v_h - w_h)$  defined in (2.3) and the nodal contact set  $\mathcal{C}_h^-(v_h - w_h)$  defined in (2.8). Lemma 4.1 (discrete Alexandroff estimate) yields

$$(4.3) \quad \sup_{\Omega} (v_h - w_h)^- \leq C \left( \sum_{x_i \in \mathcal{C}_h^-(v_h - w_h)} |\partial \Gamma(v_h - w_h)(x_i)| \right)^{1/d},$$

whence we only need to estimate  $|\partial \Gamma(v_h - w_h)(x_i)|$  for all  $x_i \in \mathcal{C}_h^-(v_h - w_h)$ . For these nodes, we easily see that

$$\partial \Gamma(v_h - w_h)(x_i) \subset \partial(v_h - w_h)(x_i).$$

Consequently, Lemma 2.3 (addition of subdifferentials) gives

$$\partial w_h(x_i) + \partial \Gamma(v_h - w_h)(x_i) \subset \partial v_h(x_i) \quad \forall x_i \in \mathcal{C}_h^-(v_h - w_h).$$

Applying Lemma 4.2 (Brunn-Minkowski inequality), we obtain

$$\begin{aligned} |\partial w_h(x_i)|^{1/d} + |\partial \Gamma(v_h - w_h)(x_i)|^{1/d} \\ \leq |\partial w_h(x_i) + \partial \Gamma(v_h - w_h)(x_i)|^{1/d} \leq |\partial v_h(x_i)|^{1/d}, \end{aligned}$$

whence

$$|\partial \Gamma(v_h - w_h)(x_i)| \leq \left( |\partial v_h(x_i)|^{1/d} - |\partial w_h(x_i)|^{1/d} \right)^d.$$

This inequality gives us the desired estimate for  $|\partial \Gamma(v_h - w_h)(x_i)|$ . In view of (4.3), adding over all  $x_i \in \mathcal{C}_h^-(v_h - w_h)$  concludes the proof.  $\square$

A direct consequence of this stability result is the maximum principle for nodal functions, which we state next.

**Corollary 4.4** (discrete maximum principle). *Let  $v_h$  and  $w_h$  be two nodal functions associated with nodes  $\mathcal{N}_h$ . If  $v_h(x_i) \geq w_h(x_i)$  at all  $x_i \in \partial\Omega$  and  $|\partial v_h(x_i)| \leq |\partial w_h(x_i)|$  at all  $x_i \in \mathcal{N}_h^0$ , then*

$$w_h(x_i) \leq v_h(x_i) \quad \forall x_i \in \mathcal{N}_h.$$

*Proof.* For any node  $x_i \in \mathcal{C}_h^-(v_h - w_h)$ , we have

$$\partial w_h(x_i) \subset \partial v_h(x_i).$$

Since  $|\partial v_h(x_i)| \leq |\partial w_h(x_i)|$  for all  $x_i \in \mathcal{N}_h^0$ , we deduce  $|\partial v_h(x_i)| = |\partial w_h(x_i)|$  for all  $x_i \in \mathcal{C}_h^-(v_h - w_h)$ . Consequently, Proposition 4.3 (stability) implies

$$\sup_{\Omega} (v_h - w_h)^- = 0,$$

whence  $v_h - w_h \geq 0$ . This completes the proof.  $\square$

## 5. CONSISTENCY

In this section, we examine the consistency of the Oliker-Prussner method (3.3). In general, this method is consistent in the sense that the right hand side of the (3.3) can be written equivalently as  $\sum_{x_i \in \mathcal{N}_h} f_i \delta_{x_i}$  and this converges to  $f$  in measure. However, such a concept of convergence is too weak to derive rates of convergence. Fortunately, we realize that if internal nodes are translation invariant, then a reasonable notion of operator consistency holds for any convex quadratic polynomial; see Lemma 5.3. Such property is shown in [4, 33] for Cartesian nodes. In contrast, we give here an alternative proof of consistency based on the geometric interpretation of subdifferentials of convex quadratic polynomials in the interior of the domain, extend the results to  $C^{2,\alpha}$  and  $W_p^s$  functions, and further investigate the consistency error in the region close to the boundary.

Lemma 2.4 (characterization of subdifferential) states that the subdifferential of a convex nodal function  $p_h$  at node  $x_i \in \mathcal{N}_h^0$  is the convex hull of piecewise constant gradients of its convex envelope  $\Gamma(p_h)$ , which in turn is determined by the nodal values  $p_h(x_j)$  in the adjacent set  $A_i(p_h)$  of  $x_i$  for  $p_h$ . The following lemma gives an estimate of the size of  $A_i(p_h)$ .

**Lemma 5.1** (size of adjacent sets). *Let the nodal set  $\mathcal{N}_h$  be quasi-uniform and shape-regular with constant  $\sigma$ . Let  $p$  be a  $C^2$  convex function defined in  $\Omega$ . If  $\lambda I \leq D^2 p \leq \Lambda I$  in  $\Omega$  for some constants  $\lambda, \Lambda > 0$  and  $p_h := N_h p$  is the nodal function associated with  $p$  defined in (2.5), then the adjacent set of nodes  $A_i(p_h)$  at  $x_i$  for  $p_h$  satisfies*

$$A_i(p_h) \subset B_{Rh}(x_i)$$

where  $R = \frac{\Lambda}{\lambda} \sigma^2$  and  $B_{Rh}(x_i)$  is the ball centered at  $x_i$  with radius  $Rh$ .

*Proof.* Let  $z$  be an adjacent node of  $x_i$  such that

$$|z - x_i| = \max\{|x_j - x_i| : x_j \in A_i(p_h)\}.$$

Without loss of generality, we may assume that  $x_i = 0$ ,  $p(x_i) = 0$  and  $\nabla p(x_i) = 0$  and set  $x_0 = x_i$ . Let  $\omega_0$  be a star at  $x_0$  in mesh  $\mathcal{T}_h^0$  associated with nodal set  $\mathcal{N}_h$ . If  $z \in \omega_0$ , then the assertion is trivial because  $R \geq 1$ .

If  $z \notin \omega_0$ , then we may assume that there is a constant  $R \geq 1$  such that  $R^{-1}z \in T$  for some element  $T \subset \omega_0$  and  $R^{-1}|z| \geq \rho_T$ . If  $\{x_k\}_{k=0}^d$  are the vertices of simplex  $T$ , then we write

$$z = R \sum_{k=0}^d \alpha_k x_k, \quad \alpha_k \geq 0, \quad \sum_{k=0}^d \alpha_k = 1.$$

We next note that  $p(x_k) \leq \frac{1}{2} \Lambda h_T^2$  for all  $1 \leq k \leq d$  because  $D^2 p \leq \Lambda I$  and  $|x_k| \leq h_T$ . Since  $z \in A_i(p_h)$ , there exists a supporting hyperplane  $L$  at  $x_0$  such that

$$L(z) = p_h(z), \quad L(x_k) \leq p_h(x_k) \leq \frac{1}{2} \Lambda h_T^2.$$



Exploiting that  $L$  is linear yields

$$p_h(z) = R \sum_{k=0}^d \alpha_k L(x_k) \leq \frac{1}{2} \Lambda h_T^2 R$$

On the other hand, since  $D^2p \geq \lambda I$  and  $|z| \geq R\rho_T = Rh_T\sigma_T^{-1}$ , we have

$$p_h(z) = p(z) \geq \frac{\lambda}{2} |z|^2 \geq \frac{\lambda}{2} R^2 \sigma_T^{-2} h_T^2.$$

Combining the last two inequalities implies

$$R \leq \frac{\Lambda}{\lambda} \sigma_T^2 \leq \frac{\Lambda}{\lambda} \sigma^2.$$

This completes the proof.  $\square$

Lemma 5.1 (size of adjacent sets) shows that  $\partial u_h(x_i)$  does not depend on values of  $u_h$  on the boundary  $\partial\Omega$  for nodes  $x_i$  such that  $\text{dist}(x_i, \partial\Omega) \geq Rh$ . We now consider nodes which are  $Rh$  away from  $\partial\Omega$  and gather several properties of subdifferential of convex quadratic polynomials.

**Lemma 5.2** (properties of convex quadratic polynomials). *Let  $p$  be a convex quadratic polynomial such that  $\lambda I \leq D^2p \leq \Lambda I$  and  $p_h := N_h p$  be the nodal function defined in (2.5). If  $R = \frac{\Lambda}{\lambda} \sigma^2$ , then the following properties hold:*

- *The subdifferential  $\partial p_h(x_i)$  is a non-empty set for all  $x_i \in \mathcal{N}_h$ .*
- *If the nodal set  $\mathcal{N}_h^0$  is translation invariant and  $\text{dist}(x_i, \partial\Omega) \geq Rh$  for  $x_i \in \mathcal{N}_h^0$ , then a uniform refinement  $\mathcal{N}_{h/2}$  of  $\mathcal{N}_h$  satisfies*

$$|\partial p_h(x_i)| = 2^d |\partial p_{\frac{h}{2}}(x_i)|.$$

- *If the nodal set  $\mathcal{N}_h^0$  is translation invariant and  $\text{dist}(x_i, \partial\Omega) \geq Rh$  and  $\text{dist}(x_j, \partial\Omega) \geq Rh$  for  $x_i, x_j \in \mathcal{N}_h^0$ , then  $A_i(p_h) = (x_i - x_j) + A_j(p_h)$  and*

$$|\partial p_h(x_i)| = |\partial p_h(x_j)|.$$

*Proof.* Take  $x_i = 0$  for simplicity. To prove the first assertion, we just observe that if  $L$  is the tangent plane touching  $p$  from below at 0, then  $L$  is a lower supporting hyperplane of  $p_h$  at 0. This implies that  $\nabla L = \nabla p(0)$  is in the subdifferential of  $p_h$  at 0.

To prove the second assertion, we consider the auxiliary polynomial

$$q^i(x) := p(x) - \nabla p(0) \cdot x - p(0),$$

obtained by subtracting the tangent plane of  $p(x)$  at 0. Since adding an affine function does not change the measure of the subdifferential, we have  $|\partial p_h(0)| = |\partial q_h^i(0)|$ . Using that  $q^i$  is homogeneous of degree 2 yields

$$q^i(x) = 4q^i\left(\frac{x}{2}\right).$$

Since

$$4q_{\frac{h}{2}}^i\left(\frac{x_j}{2}\right) = 4q^i\left(\frac{x_j}{2}\right) = q^i(x_j) = q_h^i(x_j) \geq v \cdot x_j = 2v \cdot \frac{x_j}{2}$$

for all  $x_j \in \mathcal{N}_h^0$  and  $v \in \partial q_h^i(0)$ , we deduce

$$\partial q_h^i(0) = 2\partial q_{\frac{h}{2}}^i(0),$$

whence  $|\partial q_h^i(0)| = 2^d |\partial q_{\frac{h}{2}}^i(0)|$  and the second assertion follows.

To prove the third assertion, we write  $q^i(x) = (x - x_i)^t Q (x - x_i)$  for a suitable positive semi-definite constant matrix  $Q$ . This implies

$$q^i(x) = q^j(x + x_j - x_i) \quad \forall x \in \mathbb{R}^d$$

along with

$$A_i(q_h^i) = (x_i - x_j) + A_j(q_h^j), \quad |\partial q_h^i(x_i)| = |\partial q_h^j(x_j)|.$$

Since  $A_i(p_h) = A_i(q_h^i)$  and  $\partial p_h(x_i) = \partial q_h^i(x_i)$ , this concludes the proof.  $\square$

Now we are ready to prove the consistency of (3.3).

**Lemma 5.3** (consistency of the Monge-Ampère measure). *Let  $p$  be a convex quadratic polynomial such that  $\lambda I \leq D^2 p \leq \Lambda I$  and  $p_h := N_h p$  be the corresponding convex nodal function defined in (2.5). Let  $\mathcal{N}_h^0$  be translation invariant and  $\mathcal{T}_h$  be a mesh with nodes  $\mathcal{N}_h$  and translation invariant basis of piecewise linear functions  $\{\phi_i\}_{i=1}^n$ . Then*

$$|\partial p_h(x_i)| = \int_{\Omega} \phi_i(x) \det D^2 p(x) dx$$

for any point  $x_i \in \mathcal{N}_h^0$  such that  $\text{dist}(x_i, \partial\Omega) \geq Rh$  and  $R = \frac{\Lambda}{\lambda} \sigma^2$ .

*Proof.* We consider a sequence of uniform refinements  $\mathcal{T}_k$  and corresponding nodes  $\mathcal{N}_k$  with  $h_k = 2^{-k}h$  for  $k \geq 1$ . Let  $p_k = N_{h_k} p$  be the nodal function of  $p$  associated with the set  $\mathcal{N}_k$  and let  $\gamma_k = \Gamma(p_k)$  be its convex envelope. Since  $\gamma_k \rightarrow p$  uniformly as  $k \rightarrow \infty$ , Lemma 3.1 yields

$$\int_{\Omega} \phi_i(x) dM \gamma_k(x) \rightarrow \int_{\Omega} \phi_i(x) dM p(x) \quad \text{as } k \rightarrow \infty,$$

or equivalently

$$\sum_{x_j \in \mathcal{N}_k} \phi_i(x_j) |\partial p_k(x_j)| \rightarrow \int_{\Omega} \phi_i(x) \det D^2 p(x) dx \quad \text{as } k \rightarrow \infty.$$

Therefore, we only need to prove

$$\sum_{x_j \in \mathcal{N}_k} \phi_i(x_j) |\partial p_k(x_j)| = |\partial p_h(x_i)|,$$

which is independent of  $k$ . Since  $\int_{\Omega} \phi_i^k = 2^{-kd} \int_{\Omega} \phi_i$  and  $\text{dist}(x_i, \partial\Omega) \geq Rh$ , Lemma 5.2 (properties of convex interpolation) leads to

$$|\partial p_k(x_j)| = |\partial p_k(x_i)| = |\partial p_h(x_i)| \frac{\int_{\Omega} \phi_i^k}{\int_{\Omega} \phi_i} \quad \forall k \geq 1,$$

where  $\{\phi_j^k\}$  is the basis of hat functions over  $\mathcal{T}_k$ . Consequently, we obtain

$$\sum_{x_j \in \mathcal{N}_k} \phi_i(x_j) |\partial p_k(x_j)| = \frac{|\partial p_h(x_i)|}{\int_{\Omega} \phi_i} \sum_{x_j \in \mathcal{N}_k} \phi_i(x_j) \int_{\Omega} \phi_j^k = |\partial p_h(x_i)|$$

because  $\int_{\Omega} \phi_i^k = \int_{\Omega} \phi_j^k$  according to (2.2) and  $\sum_{x_j \in \mathcal{N}_k} \phi_i(x_j) \phi_j^k = \phi_i$ , or equivalently  $\phi_i \in \text{span} \{\phi_j^k\}_{x_j \in \mathcal{N}_k}$ . This completes the proof.  $\square$

The following two local and quantitative consistency estimates of Proposition 5.4 are a consequence of Lemma 5.3 for nodes away from  $\partial\Omega$ .

**Proposition 5.4** (interior consistency). *Let  $\mathcal{N}_h^0$  be a translation invariant set of nodes,  $\mathcal{T}_h$  be a mesh with nodes  $\mathcal{N}_h$  and translation invariant basis of piecewise linear functions  $\{\phi_i\}$ . Let  $u \in C^{2,\alpha}(\overline{B_i})$  be a convex function so that  $\lambda I \leq D^2 u \leq \Lambda I$  in the ball  $B_i := B_{Rh}(x_i)$  centered at node  $x_i \in \mathcal{N}_h^0$  and radius  $Rh$  with  $R = \frac{\Lambda}{\lambda} \sigma^2$ . If  $x_i \in \mathcal{N}_h^0$  satisfies  $\text{dist}(x_i, \partial\Omega) \geq Rh$ , then*

$$\left| |\partial N_h u(x_i)| - \int_{\Omega} \phi_i(x) \det D^2 u(x) dx \right| \leq Ch^\alpha |u|_{C^{2,\alpha}(\overline{B_i})} \int_{\Omega} \phi_i(x) dx,$$

where  $C = C(d, \lambda, \Lambda)$  and  $N_h u$  denotes the convex nodal function associated with  $u$  defined in (2.5). If instead  $u(x) \in W_q^s(B_i)$  with  $s - \frac{d}{q} > 2$ ,  $s \leq 3$ , then there is again  $C = C(d, \lambda, \Lambda)$  such that

$$\left| |\partial N_h u(x_i)| - \int_{\Omega} \phi_i(x) \det D^2 u(x) dx \right| \leq Ch^{s-2-\frac{d}{q}} |u|_{W_q^s(B_i)} \int_{\Omega} \phi_i(x) dx.$$

*Proof.* To prove the first estimate we only need to show the inequality

$$|\partial N_h u(x_i)| \leq \int_{\Omega} \phi_i(x) \det D^2 u(x) dx + Ch^\alpha |u|_{C^{2,\alpha}(\overline{B_i})} \int_{\Omega} \phi_i(x) dx,$$

because the reverse inequality can be derived similarly.

Since  $u \in C^{2,\alpha}(\overline{B_i})$ , we estimate  $u$  by a quadratic polynomial  $p$  so that

$$u(x) \leq p(x) \quad \forall x \in B_{Rh}(x_i),$$

where  $p(x_i) = u(x_i)$ ,  $\nabla p(x_i) = \nabla u(x_i)$  and  $D^2 p = D^2 u(x_i) + Ch^\alpha |u|_{C^{2,\alpha}(\overline{B_i})} I$  with universal constant  $C$ . If  $p_h = N_h p$ , then Lemma 2.2 (subdifferential monotonicity) yields

$$|\partial N_h u(x_i)| \leq |\partial p_h(x_i)|.$$

If  $\phi_i$  is the hat function over  $\mathcal{T}_h$  associated with  $x_i$ , it remains to show

$$|\partial p_h(x_i)| \leq \int_{\Omega} \phi_i(x) \det D^2 u(x) dx + Ch^\alpha |u|_{C^{2,\alpha}(\overline{B_i})} \int_{\Omega} \phi_i(x) dx.$$

Since  $(\lambda + Ch^\alpha)I \leq D^2 p \leq (\Lambda + Ch^\alpha)I$  and

$$\frac{\Lambda + Ch^\alpha}{\lambda + Ch^\alpha} \leq \frac{\Lambda}{\lambda} \quad \text{because } \Lambda \geq \lambda,$$

invoking Lemma 5.3 (consistency of the Monge-Ampère measure), we obtain

$$|\partial p_h(x_i)| = \int_{\Omega} \phi_i(x) \det D^2 p(x) dx$$

because this holds for any mesh  $\mathcal{T}_h$  with nodes  $\mathcal{N}_h$  and translation invariant basis  $\{\phi_i\}_{i=1}^n$  provided  $\text{dist}(x_i, \partial\Omega) \geq Rh$ . Recalling that  $u \in C^{2,\alpha}(\overline{B_i})$ , we can write  $D^2 p = D^2 u(x) + E(x)$  for all  $x \in \overline{B_i}$ , where  $|E(x)| \leq C|u|_{C^{2,\alpha}(\overline{B_i})} h^\alpha$ . Writing  $\det D^2 p = \det D^2 u(x) \det(I + E(x) D^2 u(x)^{-1})$  and using Taylor expansion yields

$$|\partial p_h(x_i)| \leq \int_{\Omega} \phi_i(x) \det D^2 u(x) dx + Ch^\alpha |u|_{C^{2,\alpha}(\overline{B_i})} \int_{\Omega} \phi_i(x) dx,$$

and concludes the proof of the Hölder estimate. Finally, if  $u \in W_q^s(B_i)$  with  $s - \frac{d}{q} > 2$ , then we resort to the Sobolev embedding  $W_q^s(B_i) \subset C^{2,\alpha}(\overline{B_i})$  with  $0 < \alpha = s - 2 - d/q < 1$  and apply the preceding Hölder estimate.  $\square$

For nodes close to the boundary, we can no longer exploit the node translation invariance and we thus get an error of order 1. We express this fact as follows.

**Lemma 5.5** (boundary consistency). *Let  $\mathcal{T}_h$  be a mesh with nodes  $\mathcal{N}_h$  and  $\{\phi_i\}$  be a basis of piecewise linear hat functions over  $\mathcal{T}_h$ . Let  $u \in W_\infty^2(B_i)$  be a convex function with  $\lambda I \leq D^2 u \leq \Lambda I$  in the set  $B_i := B_{Rh}(x_i) \cap \Omega$  with  $R = \frac{\Lambda}{\lambda} \sigma^2$ , and let  $N_h u$  be the convex nodal function associated with  $u$ . If  $x_i \in \mathcal{N}_h^0$  is a node with  $\text{dist}(x_i, \partial\Omega) \leq Rh$ , then*

$$\left| |\partial N_h u(x_i)| - \int_{\Omega} \phi_i(x) \det D^2 u(x) dx \right| \leq C |u|_{W_\infty^2(B_i)}^d \int_{\Omega} \phi_i(x) dx,$$

where the constant  $C = C(d, \lambda, \Lambda)$ . This estimate is valid for any  $x_i \in \mathcal{N}_h^0$ .

*Proof.* We proceed as in Proposition 5.4 and only show

$$|\partial N_h u(x_i)| \leq \int_{\Omega} \phi_i(x) \det D^2 u(x) dx + C |u|_{W_\infty^2(B_i)}^d \int_{\Omega} \phi_i(x) dx.$$

Since  $\lambda I \leq D^2 u \leq \Lambda I$ , we have  $A_i(N_h u) \subset B_{Rh}(x_i) \cap \Omega$  according to Lemma 5.1 (size of adjacent sets). The  $W_\infty^2$ -regularity assumption of  $u$  gives the following estimate for the piecewise constant gradient of the convex envelope  $\Gamma(N_h u)$  of  $N_h u$  over each element  $K$  of the mesh induced by  $\Gamma(N_h u)$ , not necessarily  $\mathcal{T}_h$ , and contained in  $B_{Rh}(x_i)$

$$\nabla \Gamma(N_h u)|_K = \nabla u(x_i) + v_K, \quad |v_K| \leq Ch |u|_{W_\infty^2(B_i)}.$$

Applying Lemma 2.4 (characterization of subdifferential) we deduce that the convex hull of all  $\nabla \Gamma(N_h u)|_K$  for  $K \ni x_i$ , whence  $\partial N_h u(x_i)$ , can be bounded by a ball of radius  $Ch |u|_{W_\infty^2(B_i)}$  centered at  $\nabla u(x_i)$ . Hence, we

arrive at

$$\begin{aligned} |\partial N_h u(x_i)| &\leq C |u|_{W_\infty^2(B_i)}^d \int_{\Omega} \phi_i(x) dx \\ &\leq \int_{\Omega} \phi_i(x) \det D^2 u(x) dx + C |u|_{W_\infty^2(B_i)}^d \int_{\Omega} \phi_i(x) dx, \end{aligned}$$

because  $\det D^2 u(x) \geq 0$  a.e.  $x \in \Omega$ . This completes the proof.  $\square$

## 6. RATES OF CONVERGENCE

Our goal in this section is to establish rates of convergence in the max-norm for the approximation (3.3) of the Monge-Ampère equation (1.1). We first deal with classical solutions  $u \in C^2(\overline{\Omega})$  and next with non-classical solutions  $u \in C^{1,1}(\overline{\Omega}) \setminus C^2(\overline{\Omega})$ . Interior error estimates result from combining the stability and consistency estimates derived in Sections 4 and 5. Boundary error estimates entail a different approach involving discrete barrier functions, which we discuss next.

**6.1. Discrete Barrier function.** Since the consistency estimates of Proposition 5.4 are valid in the interior, we need to treat the boundary layer  $\{x \in \Omega : \text{dist}(x, \partial\Omega) \leq Rh\}$  differently. We exploit that  $u_h = 0$  on  $\partial\Omega$  together with the fact that  $u_h$  cannot grow too fast from  $\partial\Omega$ . This is a consequence of the next result.

**Lemma 6.1** (discrete barrier). *Let  $\Omega$  be uniformly convex and  $\mathcal{N}_h^0$  be translation invariant. Given a constant  $E > 0$ , for each node  $x_i \in \mathcal{N}_h^0$  with  $\text{dist}(x_i, \partial\Omega) \leq Rh$ ,  $R = \frac{\Lambda}{\lambda} \sigma^2$ , there exists a nodal function  $b_h^i$  such that  $|\partial b_h^i(x_j)| \geq E \int_{\Omega} \phi_j(x) dx$  for all  $x_j \in \mathcal{N}_h^0$ ,  $b_h^i(x_j) \leq 0$  at  $x_j \in \partial\Omega$  and*

$$|b_h^i(x_i)| \leq CRE^{1/d}h,$$

*provided that  $h$  is sufficiently small.*

*Proof.* We proceed in three steps. We denote  $z = x_i$  for convenience.

**Step 1.** We first construct a nodal function  $p_h$  such that

$$|\partial p_h(x_j)| \geq E \int_{\Omega} \phi_j(x) dx \quad \forall x_j \in \mathcal{N}_h^0.$$

Let  $z_0 \in \partial\Omega$  be such that  $|z - z_0| = \text{dist}(z, \partial\Omega)$ . We introduce a coordinate system with origin at  $z_0$  and  $z = (0, \dots, 0, |z - z_0|)$  and the domain  $\Omega$  lying within the sphere  $S_r$  given by

$$x_1^2 + x_2^2 + \dots + x_{d-1}^2 + (x_d - r)^2 \leq r^2,$$

where the radius  $r$  is a lower bound for the curvature of the boundary  $\partial\Omega$  which is strictly positive. Let  $p(x)$  be the convex quadratic polynomial

$$p(x) = \frac{E^{1/d}}{2} \{x_1^2 + x_2^2 + \dots + x_{d-1}^2 + (x_d - r)^2 - r^2\},$$

which is  $\leq 0$  in  $\Omega$ . We consider an extension  $\mathcal{N}_h^+$  of the nodal set  $\mathcal{N}_h^0$  to  $\mathbb{R}^d$

$$\mathcal{N}_h^+ = \left\{ z = h \sum_{j=1}^d k_j e_j : k_j \in \mathbb{Z} \right\}$$

where  $\{e_j\}$  is the basis spanning the translation invariant set  $\mathcal{N}_h^0$ . Let  $p_h^+ = N_h p$  be the nodal function associated with  $p$  over  $\mathcal{N}_h^+$ , namely  $p_h^+(x_j) = p(x_j)$  for all  $x_j \in \mathcal{N}_h^+$ . Since  $\mathcal{N}_h^+$  is translation invariant, Lemma 5.3 (consistency of the Monge-Ampère measure) yields

$$|\partial p_h^+(x_j)| = \int_{\omega_j} \phi_j(x) \det D^2 p(x) dx = E \int_{\omega_j} \phi_j(x) dx \quad \forall x_j \in \mathcal{N}_h^+,$$

where  $\omega_j = \text{supp}(\phi_j)$ . To define the nodal function  $p_h$  on  $\mathcal{N}_h$ , we set

$$p_h(x_j) := p_h^+(x_j) \quad \forall x_j \in \mathcal{N}_h^0.$$

To define  $p_h$  at boundary nodes  $x_j \in \partial\Omega$ , which may not belong to  $\mathcal{N}_h^+$ , we regard the convex envelope  $\Gamma(p_h^+)$  of  $p_h^+$  in  $\mathbb{R}^d$  as a natural extension and we assign  $p_h(x_j) := \Gamma(p_h^+)(x_j)$  for all  $x_j \in \partial\Omega$ . In view of Lemma 2.1 (discrete subdifferential), we realize that  $\partial p_h(x_j) = \partial \Gamma(p_h^+)(x_j)$  for all  $x_j \in \mathcal{N}_h^0$ .

**Step 2.** We assert that

$$p_h(x_j) \leq CE^{1/d}h \quad \forall x_j \in \mathcal{N}_h.$$

Since  $p_h(x_j) \leq 0$  for all  $x_j \in \mathcal{N}_h^0$ , we only need to show this for  $x_j \in \partial\Omega$ . For each such node, due to the convexity of  $p_h^+$ , we have

$$p_h(x_j) = \Gamma(p_h^+)(x_j) \leq \max\{p_h^+(x_k) : x_k \in A_j(p_h^+)\},$$

where  $A_j(p_h^+)$  is the adjacent set of  $x_j$  for  $p_h^+$ . By Lemma 5.1 (size of adjacent sets),  $A_j(p_h^+)$  is contained in a ball  $B_j = B_{Rh}(x_j)$ . We thus deduce

$$p_h(x_j) \leq \max\{p_h^+(x_k) : x_k \in \Omega + B_{Rh}(0)\}.$$

Since  $\Omega$  is contained in the ball  $S_r$ , we infer that

$$p_h(x_j) \leq E^{1/d} \{(r + Rh)^2 - r^2\} \leq 3E^{1/d}rRh$$

for  $h$  sufficiently small.

**Step 3.** Finally, if we set

$$b_h^i(x) := p_h(x) - 3E^{1/d}rRh,$$

then we have  $b_h^i(x_j) \leq 0$  for all  $x_j \in \partial\Omega$ . Moreover, we have

$$|b_h^i(z)| = |p(z) - 3E^{1/d}rRh| \leq CE^{1/d}rRh.$$

This completes the proof.  $\square$

**6.2. Rates of Convergence for Classical Solutions.** We prove rates of convergence for  $C^2$  classical solutions of (1.1), assuming either Hölder or Sobolev regularity of  $D^2u$ . This is the content of Theorems 6.1 and 6.2.

**Theorem 6.1** (rate of convergence for  $C^{2,\alpha}$  solutions). *Let  $\Omega$  be uniformly convex and  $\mathcal{N}_h^0$  be translation invariant. Let  $u$  be the convex solution of the Monge-Ampère equation (1.1) with  $\lambda I \leq D^2u \leq \Lambda I$  in  $\Omega$  and  $u_h$  be the solution of (3.3) with right-hand sides  $\{f_i\}_{i=1}^n$  defined over a mesh  $\mathcal{T}_h^0$  with nodes  $\mathcal{N}_h$  and translation invariant basis  $\{\phi_i\}_{i=1}^n$ . If  $f(x) \geq \lambda_F > 0$  for all  $x \in \Omega$  and  $u \in C^{2,\alpha}(\overline{\Omega})$ , then*

$$\|u - \Gamma(u_h)\|_{L^\infty(\Omega)} \leq Ch^\alpha$$

where the constant  $C = C(d, \Omega, \lambda, \Lambda, \lambda_F)(|u|_{C^{2,\alpha}(\overline{\Omega})} + |u|_{W_\infty^2(\Omega)})$ .

*Proof.* Let  $\mathcal{T}_h^0$  be the mesh with nodes  $\mathcal{N}_h$  where we define the nodal values  $f_i$  according to (3.2), and let  $N_h u$  be the nodal function associated with  $u$ . Lemma 5.1 (size of adjacent sets) gives  $A_i(N_h u) \subset B_{Rh}(x_i)$  for all  $x_i \in \mathcal{N}_h^0$ . Classical interpolation theory thus yields [9]

$$\|u - \Gamma(N_h u)\|_{L^\infty(\Omega)} \leq Ch^2 |u|_{W_\infty^2(\Omega)}.$$

Therefore, we only need to prove that  $|(N_h u - u_h)(x_i)| \leq Ch^\alpha$  for all  $x_i \in \mathcal{N}_h^0$ , or equivalently the one-sided estimate

$$(6.1) \quad \sup_{x_i \in \mathcal{N}_h} (N_h u - u_h)^-(x_i) \leq Ch^\alpha$$

because a corresponding inequality for  $(N_h u - u_h)^+$  can be derived similarly.

**Step 1** (boundary estimate). We first show that for all  $x_i \in \mathcal{N}_h^0$  such that  $\text{dist}(x_i, \partial\Omega) \leq Rh$  with  $R = \frac{\Lambda}{\lambda} \sigma^2$

$$(u_h - N_h u)(x_i) \leq C|u|_{W_\infty^2(B_i)} h.$$

Fix  $x_i \in \mathcal{N}_h^0$  and let  $b_h^i$  be the discrete barrier defined in Lemma 6.1 (discrete barrier) with free parameter  $E$ . We consider the nodal function  $u_h + b_h^i$ , which satisfies

$$\partial u_h(x_j) + \partial b_h^i(x_j) \subset \partial(u_h + b_h^i)(x_j)$$

due to Lemma 2.3 (addition of subdifferentials). Applying (4.2), which is a consequence of Lemma 4.2 (Brunn-Minkowski inequality), implies

$$|\partial(u_h + b_h^i)(x_j)| \geq |\partial u_h(x_j)| + |\partial b_h^i(x_j)|.$$

Since  $\partial u_h(x_j) = f_j$  according to (3.3), invoking Lemmas 6.1 (discrete barrier) and 5.5 (boundary consistency) yields

$$|\partial(u_h + b_h^i)(x_j)| \geq f_j + E \int_\Omega \phi_j(x) dx \geq |\partial N_h u(x_j)| \quad \forall x_j \in \mathcal{N}_h^0,$$

provided  $E \geq C|u|_{W_\infty^2(B_j)}^d$ . Moreover  $b_h^i \leq 0$  on  $\partial\Omega$  implies  $u_h + b_h^i \leq N_h u$  on  $\partial\Omega$ , whence Corollary 4.4 (discrete maximum principle) gives

$$u_h(x_j) + b_h^i(x_j) \leq N_h u(x_j) \quad \text{for all } x_j \in \mathcal{N}_h^0.$$

Finally, the estimate for  $b_h^i(x_i)$  of Lemma 6.1 (discrete barrier) yields

$$(6.2) \quad u_h(x_i) - N_h u(x_i) \leq -b_h^i(x_i) \leq C|u|_{W_\infty^2(B_i)}h.$$

**Step 2** (interior estimate). We intend to apply Proposition 4.3 (stability) to the nodal function  $N_h u - u_h$  for all nodes  $x_i \in \mathcal{N}_h^0$  with  $\text{dist}(x_i, \partial\Omega) \geq Rh$ , for which we need to compare subdifferentials and verify boundary conditions. To deal with the former, we restate Proposition 5.4 (interior consistency) with the help of (3.3), namely  $|\partial u_h(x_i)| = f_i$ :

$$|\partial N_h u(x_i)| \leq |\partial u_h(x_i)| + Ch^\alpha |u|_{C^{2,\alpha}(\overline{\Omega})} \int_{\Omega} \phi_i(x) dx.$$

Setting  $\epsilon := Ch^\alpha |u|_{C^{2,\alpha}(\overline{\Omega})} \int_{\Omega} \phi_i(x) dx$ , we readily see that

$$|\partial N_h u(x_i)|^{1/d} - |\partial u_h(x_i)|^{1/d} \leq (f_i + \epsilon)^{1/d} - f_i^{1/d}$$

for  $x_i \in \mathcal{C}_h^-(N_h u - u_h)$ . Since the function  $\psi(t) = t^{1/d}$  is concave for  $t > 0$ , we deduce that  $\psi(t + \epsilon) - \psi(t) \leq d^{-1}t^{1/d-1}\epsilon$ , whence

$$|\partial N_h u(x_i)|^{1/d} - |\partial u_h(x_i)|^{1/d} \leq Ch^\alpha |u|_{C^{2,\alpha}(\overline{\Omega})} f_i^{1/d-1} \int_{\Omega} \phi_i(x) dx.$$

Exploiting now the lower bound of  $f$ , namely  $f(x) \geq \lambda_F > 0$  for all  $x \in \Omega$ , we estimate  $f_i$  from below

$$f_i = \int_{\Omega} f(x) \phi_i(x) dx \geq \lambda_F \int_{\Omega} \phi_i(x) dx,$$

and insert this bound back into the preceding expression to obtain

$$\left( |\partial N_h u(x_i)|^{1/d} - |\partial u_h(x_i)|^{1/d} \right)^d \leq C \lambda_F^{1-d} h^{\alpha d} |u|_{C^{2,\alpha}(\overline{\Omega})}^d \int_{\Omega} \phi_i(x) dx.$$

In order to apply Proposition 4.3 (stability) to the nodal function  $N_h u - u_h$  it remains to check boundary conditions on the smaller domain

$$\Omega_h^0 := \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq Rh\}$$

where the above calculation is valid. Since  $(u_h - N_h u)(x_i) \leq C|u|_{W_\infty^2(B_i)}h$  for  $x_i \in \mathcal{N}_h^0 \setminus \Omega_h^0$ , according to (6.2), Proposition 4.3 leads to

$$\sup_{x_i \in \Omega_h^0 \cap \mathcal{N}_h^0} (N_h u - u_h + C|u|_{W_\infty^2(B_i)}h)^-(x_i) \leq C \lambda_F^{1/d-1} |\Omega|^{1/d} h^\alpha |u|_{C^{2,\alpha}(\overline{\Omega})},$$

or equivalently to

$$(6.3) \quad (u_h - N_h u)(x_i) \leq C|u|_{W_\infty^2(B_i)}h + C \lambda_F^{1/d-1} |\Omega|^{1/d} h^\alpha |u|_{C^{2,\alpha}(\overline{\Omega})}$$

for all  $x_i \in \Omega_h^0 \cap \mathcal{N}_h^0$ . Combining (6.2) with (6.3) proves the desired estimate (6.1) and concludes the proof.  $\square$



*Remark 6.1* (non-degeneracy). The lower bound  $\lambda_F$  in the non-degeneracy assumption  $f(x) \geq \lambda_F > 0$  of Theorem 6.1 may be viewed as a stability constant because  $\lambda_F^{1/d-1}$  blows up as  $\lambda_F \rightarrow 0$ . If  $f$  vanishes somewhere, then  $\lambda_F = 0$  and we have a reduced convergence rate  $h^{\alpha/d}$  because  $(f_i + \epsilon)^{1/d} - f_i^{1/d} \leq \epsilon^{1/d}$  for all  $f_i \geq 0$ .

*Remark 6.2* ( $C^{2,\alpha}$ -regularity). It is worth observing that the assumptions on  $u$  in Theorem 6.1 can be verified from assumptions on  $f$ . In fact, if  $0 < \lambda_F \leq f(x) \leq \Lambda_F$ , then  $0 < \lambda \leq D^2 u(x) \leq \Lambda$  for some constants  $\lambda, \Lambda$  [12], [30, Section 4.1]. Moreover, if  $f(x) \in C^\alpha(\overline{\Omega})$  and  $\Omega$  is of class  $C^{2,\alpha}$ , then  $u \in C^{2,\alpha}(\overline{\Omega})$  [11], [30, Section 4.3].

**Theorem 6.2** (rate of convergence for  $W_q^s$  solutions). *Let  $\Omega$  be uniformly convex and  $\mathcal{N}_h^0$  be translation invariant. Let  $u$  be the convex solution of the Monge-Ampère equation (1.1) with  $\lambda I \leq D^2 u \leq \Lambda I$  in  $\Omega$  and  $u_h$  be the solution of (3.3) with right-hand sides  $\{f_i\}_{i=1}^n$  defined over a mesh  $\mathcal{T}_h^0$  with nodes  $\mathcal{N}_h$  and translation invariant basis  $\{\phi_i\}_{i=1}^n$ . If  $f(x) \geq \lambda_F > 0$  for all  $x \in \Omega$  and  $u \in W_q^s(\Omega)$  with  $s > 2 + \frac{d}{q}$ ,  $s \leq 3$ ,  $d < q \leq \infty$ , then*

$$\|u - \Gamma(u_h)\|_{L^\infty(\Omega)} \leq Ch^{s-2}$$

where the constant  $C = C(d, \Omega, \lambda, \Lambda, \lambda_F)(|u|_{W_q^s(\Omega)} + |u|_{W_\infty^2(\Omega)})$ .

*Proof.* We just employ the second estimate of Proposition 4.3 (interior consistency) in Step 2 of the proof of Theorem 6.1 (rate of convergence for  $C^{2,\alpha}$  solutions). The key difference is the estimate of the sum

$$S := h^{s-2-\frac{d}{q}} \left( \sum_{x_i \in \mathcal{N}_h^0} |u|_{W_q^s(B_i)}^d \int_{\Omega} \phi_i(x) dx \right)^{\frac{1}{d}}$$

before applying Proposition 4.3 (stability); recall that  $B_i = B(x_i, Rh)$ . To exploit the  $\ell^q$ -summability of  $\{|u|_{W_q^s(B_i)}\}_{x_i \in \mathcal{N}_h^0}$  we utilize Hölder inequality with exponents  $t = q/d$  and  $t^* = q/(q-d)$  to arrive at

$$S \leq h^{s-2-\frac{d}{q}} \left( \sum_{x_i \in \mathcal{N}_h^0} |u|_{W_q^s(B_i)}^q \right)^{\frac{1}{q}} \left( \sum_{x_i \in \mathcal{N}_h^0} \left( \int_{\Omega} \phi_i \right)^{\frac{q}{q-d}} \right)^{\frac{q-d}{dq}}.$$

We note that the balls  $B_i$  have a finite overlapping property. In fact, if  $B_i \cap B_j \neq \emptyset$ , then  $\text{dist}(x_i, x_j) \leq 2Rh$  and the box

$$(6.4) \quad \omega_j := \left\{ x = x_j + \sum_{k=1}^d t e_k : -\frac{h}{2} \leq t \leq \frac{h}{2} \right\}$$

is contained in  $B_{Ch}(x_i)$  where  $C = 2R + \frac{1}{2}d$ . Since  $|B_{Ch}(x_i)| \leq Ch^d$  and  $|\omega_j| \geq ch^d$ , with  $c = c(\sigma)$ , we deduce that balls  $B_{Ch}(x_i)$  contain a fixed

number of nodes  $x_j$  and  $B_i$  overlaps with a fixed number of  $B_j$ 's. Hence

$$\left( \sum_{x_i \in \mathcal{N}_h^0} \left( \int_{\Omega} \phi_i \right)^{\frac{q}{q-d}} \right)^{\frac{q-d}{d}} \leq C |\Omega|^{\frac{q-d}{dq}} h^{\frac{d}{q}},$$

and we deduce that  $S \leq Ch^{s-2} |u|_{W_q^s(\Omega)}$  to conclude the proof.  $\square$

We stress that Theorem 6.2 (rate of convergence for  $W_q^s$  solutions) provides a linear rate for solutions  $u \in W_q^3(\Omega)$  with  $q > d$ , which is much weaker than the requirement  $u \in W_{\infty}^3(\Omega)$  of Theorem 6.1 (rate of convergence for  $C^{2,\alpha}$  solutions) for a similar rate. We explore this further below.

**6.3. Rates of Convergence for Piecewise Smooth Solutions.** We now study the case where the consistency error may be large in a small region, for instance when the Hessian  $D^2u$  jumps across a surface within  $\Omega$ , but is small otherwise. We therefore give up the assumption  $u \in C^2(\overline{\Omega})$ . We exploit the structure of the estimate in Proposition 4.3 (stability), namely the fact that its right-hand side accumulates in  $\ell_d$ .

Before stating our result, we introduce the Minkowski-Bouligand dimension of a subset  $\omega$  of  $\Omega$ . Given a translation invariant set of nodes  $\mathcal{N}_h = \{x_i\}$ , let  $\{\omega_i\}_{x_i \in \mathcal{N}_h}$  be the translation invariant partition covering  $\Omega$  defined in (6.4). Let  $m(h)$  be the number of  $\omega_i$ 's required to cover  $\omega$ . We define the (Minkowski-Bouligand or box) dimension of  $\omega$  to be

$$\dim \omega := - \lim_{h \rightarrow 0} \frac{\log m(h)}{\log h}.$$

For example, it is easy to check that  $\partial B_1$ , the discontinuity set of  $D^2u$  in example (3.1), is of dimension one. In addition, the solution  $u$  satisfies  $u \in W_{\infty}^2(B_2(0)) \setminus C^2(\overline{B_2(0)})$  and  $u \in C^3(\overline{B_1(0)}), C^3(\overline{B_2(0)} \setminus B_1(0))$ . The following theorem explores situations such as this one.

**Theorem 6.3** (convergence rates for piecewise smooth Hessians). *Let  $\Omega$  be uniformly convex and  $\mathcal{N}_h$  be a translation invariant set of nodes. Let  $u \in W_{\infty}^2(\Omega)$  be the convex solution of the Monge-Ampère equation (1.1) and satisfy  $\lambda I \leq D^2u \leq \Lambda I$  in  $\Omega$ . Let  $D^2u$  be piecewise smooth in the sense that  $D^2u \in W_q^s(\Omega \setminus \omega)$  with  $s > 2 + d/q$ ,  $s \leq 3$ ,  $d < q \leq \infty$ , and let  $\omega$  have box dimension  $n < d$ . If  $f \geq \lambda_F > 0$  in  $\Omega$  and  $u_h$  is the solution of (3.3) with right-hand sides  $\{f_i\}_{i=1}^n$  defined over a mesh  $\mathcal{T}_h^0$  with nodes  $\mathcal{N}_h$  and translation invariant basis  $\{\phi_i\}_{i=1}^n$ , then*

$$\|u - \Gamma(u_h)\|_{L^{\infty}(\Omega)} \leq Ch^{s-2} |u|_{W_q^s(\Omega \setminus \omega)} + Ch^{\frac{d-n}{d}} |u|_{W_{\infty}^2(\Omega)},$$

where the constant  $C = C(d, \Omega, \omega, \lambda, \Lambda, \lambda_F)$ .

*Proof.* We argue as in Theorems 6.1 and 6.2. Therefore, we first observe that  $u \in W_{\infty}^2(\Omega)$  guarantees

$$|(N_h u - u_h)(x_i)| \leq C |u|_{W_{\infty}^2(\Omega)} h$$

for all  $x_i \in \mathcal{N}_h^0$  such that  $\text{dist}(x_i, \partial\Omega) < Rh$ . We split the nodes  $x_i \in \mathcal{N}_h^0$  such that  $\text{dist}(x_i, \partial\Omega) \geq Rh$  into two sets, those that are at distance  $Rh$  to  $\omega$ , denoted by  $\mathcal{N}_h^0(\omega)$ , and the complement  $\mathcal{N}_h^0(\Omega \setminus \omega)$ .

In order to apply Proposition 4.3 (stability), we start with the estimate

$$|\partial N_h u(x_i)|^{1/d} - |\partial u_h(x_i)|^{1/d} \leq (f_i + \epsilon_i)^{1/d} - f_i^{1/d} \leq d^{-1} \lambda_F^{1/d-1} \delta_i$$

and  $\delta_i = \epsilon_i \left( \int_{\Omega} \phi_i \right)^{1/d-1}$ , in conjunction with the expression of  $\epsilon_i$  already derived in the second estimate of Proposition 5.4 (interior consistency), to arrive at

$$\delta_i = Ch^{s-2-\frac{d}{q}} |u|_{W_q^s(B_i)} \left( \int_{\Omega} \phi_i \right)^{\frac{1}{d}} \quad \forall x_i \in \mathcal{N}_h^0(\Omega \setminus \omega).$$

We next resort to the crude bound  $(f_i + \epsilon_i)^{1/d} - f_i^{1/d} \leq \epsilon_i^{1/d}$ , together with the expression of  $\epsilon_i$  derived in Lemma 5.5 (boundary consistency), to write

$$\epsilon_i = C |u|_{W_{\infty}^2(B_i)} \left( \int_{\Omega} \phi_i \right)^{\frac{1}{d}} \quad \forall x_i \in \mathcal{N}_h^0(\omega).$$

We point out that Lemma 5.5 is valid for any  $x_i \in \mathcal{N}_h^0$ . Arguing as in Theorem 6.2 (rate of convergence for  $W_q^s$  solutions), we thus obtain

$$\sum_{x_i \in \mathcal{N}_h^0(\Omega \setminus \omega)} \delta_i^d \leq Ch^{(s-2)d} |u|_{W_q^s(\Omega \setminus \omega)}^d,$$

as well as

$$\sum_{x_i \in \mathcal{N}_h^0(\omega)} \epsilon_i^d \leq C |u|_{W_{\infty}^2(\Omega)}^d \sum_{x_i \in \mathcal{N}_h^0(\omega)} |\omega_i| \leq Ch^{d-n} |u|_{W_{\infty}^2(\Omega)}^d$$

because  $\sum_{x_i \in \mathcal{N}_h^0(\omega)} |\omega_i| \leq Cm(h)h^d \leq h^{d-n}$  with  $n$  being the box dimension of  $\omega$ . Applying now Proposition 4.3 (stability) to  $N_h(u) - u_h$  yields

$$\sup (N_h(u) - u_h)^- \leq Ch^{s-2} |u|_{W_q^s(\Omega \setminus \omega)} + Ch^{\frac{d-n}{d}} |u|_{W_{\infty}^2(\Omega)}.$$

This is the asserted estimate.  $\square$

We conclude with a simple application of Theorem 6.3 (convergence rates for piecewise smooth Hessians) to the example (3.1). Since  $d = 2$ ,  $s = 3$ ,  $q = \infty$ , and  $n = \dim(\partial B_1) = 1$ , we deduce

$$\|u - \Gamma(u_h)\|_{L^{\infty}(\Omega)} \leq C(u)h^{1/2}.$$

We point out that the singular set  $\omega = \partial B_1$  need not be matched by either the original mesh  $\mathcal{T}_h^0$  or the mesh  $\mathcal{T}_h$  induced by the convex envelope  $\Gamma(u_h)$  of  $u_h$ .

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DEPARTMENT OF MATHEMATICS AND INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY, UNIVERSITY OF MARYLAND, COLLEGE PARK  
*E-mail address:* `rhn@math.umd.edu`

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY  
*E-mail address:* `wujun@math.rutgers.edu`